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Intrinsical Randomness of Kolmogorov $\mathbb{Z}^d$-Actions on a Lebesgue Space

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We introduce a concept of an intrinsically weak and strong randomness of a $\mathbb{Z}^d$-action on a Lebesgue space and we show that Kolmogorov $\mathbb{Z}^d$-actions are intrinsically weak random, and Bernoulli $\mathbb{Z}^d$-actions are intrinsically strong random.

KEY WORDS: Intrinsically weak randomness; intrinsically strong randomness; Kolmogorov $\mathbb{Z}^d$-actions; Bernoulli $\mathbb{Z}^d$-actions.

INTRODUCTION

The concept of an intrinsical randomness for one-dimensional dynamical systems (actions of the group $\mathbb{Z}$ on a Lebesgue space) has been introduced by several authors (see ref. 2 and references therein).

The intrinsically random systems are conjugated via a Markovian operator with a non-invertible semigroup of Markovian operators which monotonically converges to equilibrium.

It is shown in ref. 2 that all Kolmogorov systems are intrinsically random.

Our aim is to consider a multidimensional analogue of the concept of the intrinsical randomness.

In this paper, we define concepts of an intrinsically weak randomness (IWR) and an intrinsically strong randomness (ISR) of a $\mathbb{Z}^d$-action. These definitions contain, apart from the direct extension of the one-dimensional properties, a continuity condition, which has no corresponding property in

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the one-dimensional case. It is connected with the fact that the group $\mathbb{Z}^d$ equipped with the lexicographical order has gaps.

First we show that IWR $\mathbb{Z}^d$-actions are weakly mixing and ISR $\mathbb{Z}^d$-actions are not rigid.

Our main result says that all Kolmogorov (Bernoulli) $\mathbb{Z}^d$-actions satisfy the IWR (ISR) condition. If we interpret a $\mathbb{Z}^d$-action on a Lebesgue space as a temporal dynamical system with $d-1$ symmetries, then an IWR (ISR) $\mathbb{Z}^d$-action is conjugated to a semigroup of Markovian operators which converges to equilibrium both along time evolution and the action of the symmetries.

RESULT

Let $(X, \mathscr{B}, \mu)$ be a Lebesgue probability space and let $\mathscr{N}$ denote the trivial sub-σ-algebra of $\mathscr{B}$.

We denote by $\mathbb{Z}^d$ the group of d-dimensional integers. Let $\prec$ be the lexicographical order of $\mathbb{Z}^d$ and let $\Pi(N)$ stand for the set of positive (negative) vectors of $\mathbb{Z}^d$ with respect to $\prec$.

Let Φ be a $\mathbb{Z}^d$-action on $(X, \mathscr{B}, \mu)$, i.e., Φ is a homomorphism of $\mathbb{Z}^d$ into the group $\mathrm{Aut}(X, \mu)$ of all measure-preserving automorphisms of $(X, \mathscr{B}, \mu)$.

We denote by Φ^g the automorphism of $(X, \mathscr{B}, \mu)$, being the image of $g \in \mathbb{Z}^d$ under Φ.

Let $U = U_\Phi$ be the unitary representation of $\mathbb{Z}^d$ in $L^2(X, \mu)$ defined by the formula

$$U^g f = f \circ \Phi^g, \qquad f \in L^2(X, \mu), \qquad g \in \mathbb{Z}^d$$

i.e., U acts by the Koopman operators associated with the automorphisms Φ^g, $g \in \mathbb{Z}^d$.

For a given σ-algebra $\mathscr{A} \subset \mathscr{B}$ and a function $f \in L^1(X, \mu)$, we denote by $E^{\mathscr{A}} f$ the conditional expectation of f given $\mathscr{A}$. In particular, we put

$$Ef = E^{\mathscr{N}} f = \int_X f \, d\mu$$

We apply in the sequel the following well known property of conditional expectations

$$U_\varphi \circ E^{\mathscr{A}} = E^{\varphi^{-1}\mathscr{A}} \circ U_\varphi \tag{1}$$

where φ is a given measure-preserving automorphism of $(X, \mathscr{B}, \mu)$ and U_φ is the Koopman unitary operator associated with φ.

For a given finite measurable partition P of X, we denote by $h(P, \Phi)$ the mean entropy of P with respect to Φ and by $h(\Phi)$ the entropy of Φ.[1, 7]

Now we recall the definition of a Kolmogorov action (K-action) of $\mathbb{Z}^d$ on a Lebesgue space.[1, 5]

An ordered pair (A, B) of subsets of $\mathbb{Z}^d$ is said to be a cut if they form a non-trivial partition of $\mathbb{Z}^d$ and for every $g \in A$ and $h \in B$ we have $g \prec h$.

A cut (A, B) is called a gap if A does not contain the greatest element and B does not contain the lowest element.

Definition 1. A $\mathbb{Z}^d$-action Φ is said to be a K-action if there exists a σ-algebra $\mathscr{A} \subset \mathscr{B}$ with

(i) $\Phi^g \mathscr{A} \subset \mathscr{A}$ for every $g \in \Pi$,

(ii) the family $(\Phi^g \mathscr{A})$, $g \in \mathbb{Z}^d$ is continuous, i.e., for every gap (A, B) of $\mathbb{Z}^d$ it holds

$$\bigvee_{g \in A} \Phi^g \mathscr{A} = \bigcap_{g \in B} \Phi^g \mathscr{A}$$

(iii) $\bigvee_{g \in \mathbb{Z}^d} \Phi^g \mathscr{A} = \mathscr{B}$,

(iv) $\bigcap_{g \in \mathbb{Z}^d} \Phi^g \mathscr{A} = \mathscr{N}$.

It has been shown in ref. 5 that Φ is a K-action iff Φ has a completely positive entropy, i.e., $h(P, \Phi) > 0$ for every non-trivial finite measurable partition P of X.

The well-known examples of K-actions of $\mathbb{Z}^d$ are Bernoullian actions and Gaussian actions with absolutely continous spectral measures.

Goldstein has shown [4, Theorem 6.3] that every Poisson system of periodic σ-K-type (and in particular the Lorentz gas) is a K-action.

For simplicity of notations, we restrict ourselves in the sequel to the case $d = 2$. All our considerations may be easily extended to the general case.

A $\mathbb{Z}^2$-action Φ is uniquely determined by the automorphisms $T = \Phi^{(1, 0)}$ and $S = \Phi^{(0, 1)}$. We call them the standard automorphisms defined by Φ.

It is easy to observe that the definition of a K-action may be written by the use of these automorphisms in the following way:

(i′) $S^{-1} \mathscr{A} \subset \mathscr{A}$, $T^{-1} \mathscr{A}_S \subset \mathscr{A}$

where

$$\mathcal{A}_S = \bigvee_{n=-\infty}^{+\infty} S^n \mathcal{A}$$

(ii′) $\bigcap_{n=-\infty}^{\infty} S^n \mathcal{A} = T^{-1} \mathcal{A}_S$,

(iii′) $\bigvee_{n=-\infty}^{+\infty} T^n \mathcal{A}_S = \mathcal{B}$,

(iv′) $\bigcap_{n=-\infty}^{+\infty} T^n \mathcal{A}_S = \mathcal{N}$.

A linear operator W of $L^2(X, \mu)$ is said to be doubly stochastic if it is positive, $W1 = 1$ and $E \circ W = E$, i.e., if it is a Markov operator preserving μ.

We shall consider in the sequel weak (w) and strong (s) limits of functions defined on Π, and taking values in some Banach space.

We write

$$f(g) \xrightarrow{w} x \qquad \text{as} \quad g \longrightarrow \infty$$

if for any $\varepsilon > 0$ there exists $g_o \in \Pi$ with $\|f(g) - x\| < \varepsilon$ for all $g \succ g_o$.

On the other hand the notation

$$f(g) \xrightarrow{s} x \qquad \text{as} \quad g \longrightarrow \infty$$

means that for any $\varepsilon > 0$, there exists a finite subset $F \subset \Pi$ such that $\|f(g) - x\| < \varepsilon$ for all $g \in \Pi \backslash F$.

It is clear that

$$f(g) \xrightarrow{s} x \Rightarrow f(g) \xrightarrow{w} x \qquad \text{as} \quad g \longrightarrow \infty$$

Definition 2. A $\mathbb{Z}^2$-action Φ is called intrinsically weak (strong) (IWR (ISR)) random if there exists a doubly stochastic operator $\Lambda \neq E$ and a semigroup $(W^g, g \in \Pi)$ of doubly stochastic operators of $L^2(X, \mu)$ such that

(a) for every $g \in \Pi$, the following diagram commutes

$$\begin{array}{ccc} L^2(X,\mu) & \xrightarrow{\Lambda} & L^2(X,\mu) \\ {\scriptstyle U^{-g}}\downarrow & & \downarrow{\scriptstyle W^g} \\ L^2(X,\mu) & \xrightarrow[\Lambda]{} & L^2(X,\mu) \end{array}$$

(b) the function $g \to \| W_g f \|$, $g \in \mathbb{Z}^2$ is non-increasing and continuous for every $f \in L^2(X, \mu)$, i.e.,

$$\inf_{g \in A} \| W^g f \| = \sup_{g \in B} \| W^g f \|$$

for every gap (A, B) of $\mathbb{Z}^2$,

(c) for every $f \in L^2(X, \mu)$ with $f \geqslant 0$ and $Ef = 1$ the sequence $(\| W^g f - 1 \|,\ g \in \Pi)$ is non-increasing and

$$W^g f \xrightarrow{w} 1 (W^g f \xrightarrow{s} 1)$$

when $g \to \infty$.

It is obvious that every ISR-action is also an IWR-action.

First, we shall give a necessary condition for a $\mathbb{Z}^2$-action to be IWR (ISR).

A $\mathbb{Z}^2$-action Φ is said to be weakly mixing if for every $A, B \in \mathscr{B}$

$$\lim_{n \to \infty} \frac{1}{n^2} \sum_{g \in R_n} |\mu(\Phi^g A \cap B) - \mu(A)\, \mu(B)| = 0$$

where

$$R_n = \{(p, q) \in \mathbb{Z}^2; 0 \leqslant p, q \leqslant n - 1\}, \qquad n \geqslant 1$$

Similarly as in the case of $\mathbb{Z}$-actions one shows that Φ is weakly mixing iff the equality

$$U^{(m,n)} f = \lambda_1^m \lambda_2^n f, \quad (m, n) \in \mathbb{Z}^2, \quad |\lambda_1| = |\lambda_2| = 1, \quad f \in L^2(X, \mu), \quad f \neq 0$$

implies that f is a constant a.e.

Proposition 1. Every IWR $\mathbb{Z}^2$-action is weakly mixing.

Proof. Let $f \in L^2(X, \mu)$, $f \neq 0$ and $\lambda_1, \lambda_2 \in \mathbb{T}$ be such that

$$U^{(m,n)} f = \lambda_1^m \lambda_2^n f, \qquad (m, n) \in \mathbb{Z}^2$$

It follows from (a) that

$$W^{(m,n)} \Lambda f = \Lambda U^{-(m,n)} f = \bar{\lambda}_1^m \bar{\lambda}_2^n f, \qquad (m, n) \in \Pi$$

If Φ is IWR then

$$W^{(m,n)} \Lambda f \xrightarrow{w} E\Lambda f = Ef$$

i.e.,

$$\bar{\lambda}_1^m \bar{\lambda}_2^n f \xrightarrow{w} Ef$$

as $(m, n) \to \infty$.

But this is possible only in the case $\lambda_1 = \lambda_2 = 1$ and so $f = Ef$ a.e. This means that Φ is weakly mixing.

Remark 1. This result is to compare with a result of Misra[8] which shows that an intrinsically random $\mathbb{Z}$-action is necessarily mixing.

Now we recall the concept of a rigid $\mathbb{Z}^2$-action.

A $\mathbb{Z}^2$-action Φ is said to be rigid[3] if there exists a sequence $(m_k, n_k) \subset \mathbb{Z}^2$ such that $(m_k, n_k) \to \infty$ (i.e., $|m_k| + |n_k| \to \infty$) and

$$\lim_{k \to \infty} \mu(\Phi^{(m_k, n_k)} A \triangle A) = 0$$

for every $A \in \mathscr{B}$.

It is easy to show that Φ is rigid, iff for some sequence $(m_k, n_k) \to \infty$ and for every $f \in L^2(X, \mu)$,

$$\lim_{k \to \infty} U^{(m_k, n_k)} f = f$$

Simple examples of rigid actions are actions with discrete spectrum and a class of Gaussian actions with a singular spectral measure.[3, 6]

Proposition 2. If a $\mathbb{Z}^2$-action is ISR, then it is weakly mixing and not rigid.

Proof. The first part of Proposition 2 follows at once from Proposition 1.

Let us now suppose Φ is ISR and rigid. Let (m_k, n_k) be a sequence in $\mathbb{Z}^2$ such that $(m_k, n_k) \to \infty$ and $U^{(m_k, n_k)} f \to f$ for every $f \in L^2(X, \mu)$. One can assume that $(m_k, n_k) \in \Pi$, $k \geqslant 1$. It follows from (a) that

$$W^{(m_k, n_k)} \circ \Lambda = \Lambda \circ U^{-(m_k, n_k)}, \qquad k \geqslant 1$$

Hence, taking the limit as $k \to \infty$, one obtains

$$\lim_{k \to \infty} W^{(m_k, n_k)}(\Lambda f) = \Lambda f$$

On the other hand the property (c) implies that

$$\lim_{k \to \infty} W^{(m_k, n_k)}(\Lambda f) = E\Lambda f = Ef$$

i.e., $\Lambda = E$ which is impossible.

Now we shall show our main results.

Theorem 1. Every K-action of $\mathbb{Z}^2$ on a Lebesgue space is IWR.

Proof. Let Φ be a K-action of $\mathbb{Z}^2$, U the associated unitary representation in $L^2(X, \mu)$ and $\mathscr{A}$ the σ-algebra satisfying (i)–(iv).

We put

$$\Lambda = E^{\mathscr{A}}, \qquad W^g = \Lambda \circ U^{-g}, \qquad g \in \mathbb{Z}^2$$

It is clear that Λ and W^g, $g \in \mathbb{Z}^2$ are doubly stochastic and $\Lambda \neq E$.
Let now $g = (m, n) \in \Pi$. First we shall show that

$$\Lambda \circ U^{-g} = W^g \circ \Lambda \tag{2}$$

Let $\mathscr{A}_g = \Phi^g \mathscr{A}$. It follows from (1) that

$$U^{-g} \circ E^{\mathscr{A}} \circ U^g = E^{\mathscr{A}_g} \tag{3}$$

Indeed, if T and S denote the standard automorphisms of Φ then (1) implies

$$\begin{aligned} U^{-g} \circ E^{\mathscr{A}} &= U_T^{-m} \circ U_S^{-n} \circ E^{\mathscr{A}} = U_{T^{-m}} \circ U_{S^{-n}} \circ E^{\mathscr{A}} \\ &= U_{T^{-m}} \circ E^{S^n \mathscr{A}} \circ U_{S^{-n}} \\ &= E^{T^m S^n \mathscr{A}} \circ U_{T^{-m}} \circ U_{S^{-n}} = E^{\mathscr{A}_g} U^{-g} \end{aligned}$$

Since $g \in \Pi$ the invariance of $\mathscr{A}$ gives

$$E^{\mathscr{A}} \circ E^{\mathscr{A}_g} = E^{\mathscr{A}}$$

Combining this equality with (3) we get (2).

The equality (2) implies that the operators W^g, $g \in \Pi$ form a semigroup. Indeed, since the order $\succ$ is compatible with the group operation in $\mathbb{Z}^2$ we have

$$\begin{aligned} W^{g+g'} &= \Lambda \circ U^{-(g+g')} = \Lambda \circ U^{-g} \circ U^{-g'} \\ &= W^g \circ \Lambda \circ U^{-g'} = W^g \circ W^{g'} \end{aligned}$$

$g, g' \in \Pi$.

Now we shall check (b). It is enough to prove that

$$\lim_{n\to+\infty} \| W^{(0,n)} f \| = \lim_{n\to-\infty} \| W^{(1,n)} f \|, \qquad f\in L^2(X,\mu)$$

Applying (1) we have

$$\| W^{0,n)} f \| = \| E^{\mathscr{A}} \circ U_S^{-n} f \| = \| U_S^{-n} E^{S^{-n}\mathscr{A}} f \| = \| E^{S^{-n}\mathscr{A}} f \|$$

and similarly

$$\| W^{(1,n)} f \| = \| U_T^{-1} U_S^{-n} E^{S^{-n}T^{-1}\mathscr{A}} f \| = \| E^{T^{-1}S^{-n}\mathscr{A}} f \|$$

Taking the limit in the above two equalities and applying the condition (ii′) we obtain (b).

In order to prove (c) we take a function $f\in L^2(X,\mu)$ such that $f\geqslant 0$ and $Ef=1$.

It follows from (3) that

$$\begin{aligned} \| W^g f - 1 \|^2 &= \| W^g f \|^2 - 2E(W^g f) + 1 \\ &= \| E^{\mathscr{A}} \circ U^{-g} f \|^2 - 1 \\ &= \| U^g \circ E^{\mathscr{A}} \circ U^{-g} f \|^2 - 1 \\ &= \| E^{\mathscr{A}_{-g}} f \|^2 - 1, \qquad g\in \Pi \end{aligned} \tag{4}$$

If $g, g'\in \Pi$ and $g \prec g'$ then $\mathscr{A}_{-g} \supset \mathscr{A}_{-g'}$ and so

$$\| E^{\mathscr{A}_{-g}} f \| \geqslant \| E^{\mathscr{A}_{-g'}} f \|$$

i.e., the sequence $(\| W^g f - 1 \|,\ g\in\Pi)$ is non-increasing.

It follows from (iv) that

$$\bigcap_{g\in\Pi} \mathscr{A}_{-g} = \bigcap_{m=0}^{\infty} T^{-m}\mathscr{A}_S = \mathscr{N}$$

Let $\varepsilon>0$ be arbitrary. By (4) and the Doob martingale convergence theorem there exists $m_o\in\mathbb{N}$ such that, for every $m>m_o$ and $n\in\mathbb{Z}$, we have

$$\begin{aligned} \| W^{(m,n)} f - 1 \|^2 &= \| E^{T^{-m}S^{-n}\mathscr{A}} f \|^2 - 1 \\ &\leqslant \| E^{T^{-m}\mathscr{A}_S} f \|^2 - 1 \\ &< \varepsilon^2 \end{aligned}$$

Hence it is clear that for $g=(m,n)\succ(m_o,n_o)$, where n_o is arbitrary, we have $\|W^g f-1\|<\varepsilon$, i.e., $W^g f \xrightarrow{w} 1$ as $g\to\infty$.

We shall show in the following example that there it may exist semigroups of doubly stochastic operators which satisfy (c) but don't satisfy (b).

Example. Let X be the infinite-dimensional torus $\mathbb{T}^{\mathbb{Z}}$ equipped with the product σ-algebra $\mathscr{B}$, μ be the product measure on X determined by the Lebesgue measure λ on $\mathbb{T}$ and let φ be an irrational rotation on $\mathbb{T}$.

Let $\Phi=\Phi_\varphi$ be a $\mathbb{Z}^2$-action on X generated by the following two measure-preserving automorphisms T and $S=S_\varphi$ of X defined by

$$(Tx)(n)=x(n+1), \qquad (Sx)(n)=\varphi x(n), \qquad n\in\mathbb{Z}$$

i.e.,

$$\Phi^g=T^m\circ S^n, \qquad (m,n)\in\mathbb{Z}^2$$

Let Q be a two-element generating partition for φ and let P be the partition of X induced by Q via the projection $\pi_0\colon X\to\mathbb{T}$ on the zero coordinate, i.e., $P=\pi_0^{-1}(Q)$.

Let $\mathscr{A}$ be the smallest σ-algebra containing the partitions $\Phi^{(m,n)}P$, $(m,n)\in\Pi$.

It is easy to see that

$$\mathscr{A}=P\vee P_S^-\vee(P_S)_T^-$$

where

$$P_S^-=\bigvee_{n=1}^{\infty}S^{-n}P, \qquad P_S=\bigvee_{n=-\infty}^{+\infty}S^nP, \qquad (P_S)_T^-=\bigvee_{m=1}^{\infty}T^{-m}P_S$$

It is clear that $\mathscr{A}$ is invariant and

$$\bigvee_{g\in\mathbb{Z}^2}\Phi^g\mathscr{A}=\mathscr{B}$$

Moreover, in our case $S^{-1}\mathscr{A}=\mathscr{A}$. Indeed, since φ is a rotation the smallest σ-algebra containing the partitions $\varphi^{-n}Q$, $n\geqslant 1$ coincides with the σ-algebra $\mathscr{F}$ of the Lebesgue measurable sets of $\mathbb{T}$. Hence $P\vee P_S^-=P_S$ and so $\mathscr{A}=P_S\vee(P_S)_T^-=S^{-1}\mathscr{A}$. Since μ is the product measure the

σ-algebras $T^n P_S$, $n \in \mathbb{Z}$ are independent. Therefore the Kolmogorov zero-one law implies

$$\bigcap_{g \in \mathbb{Z}^2} \Phi^g \mathscr{A} = \bigcap_{n=0}^{\infty} T^{-n}(P_S)_T^- = \mathscr{N}$$

We put $W^g = E^{\mathscr{A}} \circ U^{-g}$, $g \in \Pi$. One checks that (W^g) is a semigroup of doubly stochastic operators of $L^2(X, \mu)$ satisfying (c) in the same way as in the proof of Theorem 1. Now we shall show that (b) is not satisfied.

Let $f_0(z) = z$, $z \in \mathbb{T}$. The independence of P_S and $(P_S)_T^-$ gives

$$\begin{aligned} \| W^{(0,n)} f_0 \circ \pi_0 \| &= \| E^{S^{-n}\mathscr{A}} f_0 \circ \pi_0 \| \\ &= \| E^{P_S \vee (P_S)_T^-} f_0 \circ \pi_0 \| \\ &= \| E^{P_S} f_0 \circ \pi_0 \| = \| f_0 \circ \pi_0 \| \\ &= \int_{\mathbb{T}} |z| \, \lambda(dz) = 1 \end{aligned}$$

On the other hand

$$\begin{aligned} \lim_{n \to \infty} \| W^{(1,n)} f \| &= \| E^{T^{-1}\mathscr{A}_S} f_0 \circ \pi_0 \| \\ &= \| E^{T^{-1} P_S} f_0 \circ \pi_0 \| = 0 \end{aligned}$$

i.e., (b) does not hold.

Let now Φ be a Bernoulli $\mathbb{Z}^2$-action on $(X, \mathscr{B}, \mu)$ with finite entropy and let P be a finite measurable partition of X such that the partitions $\Phi^g P$, $g \in \mathbb{Z}^2$ are independent and generate $\mathscr{B}$.

Theorem 2. Any Bernoulli $\mathbb{Z}^2$-action is ISR.

Proof. Let $\mathscr{A}$ be the past σ-algebra generated by P, i.e., $\mathscr{A}$ is the smallest σ-algebra containing P and the partitions $\Phi^g P$, $g \in N$. Applying the standard automorphisms T and S of Φ we may write $\mathscr{A}$ in the form

$$\mathscr{A} = P \vee P_S^- \vee (P_S)_T^-$$

where

$$P_S^- = \bigvee_{n=1}^{\infty} S^{-n} P, \qquad P_S = \bigvee_{n=-\infty}^{+\infty} S^n P$$

It is well known[5] that $\mathscr{A}$ satisfies the conditions (i′)–(iv′). Let W^g, $g \in P$ be the doubly stochastic operator defined in the proof of Theorem 1.

It is enough to show that

$$W^g f \xrightarrow{s} 1 \qquad \text{as} \quad g \longrightarrow \infty$$

for every $f \in L^2(X, \mu)$, $f \geqslant 0$ and $Ef = 1$. We denote by $\mathscr{B}_n$ the σ-algebra generated by all partitions of the form $\Phi^h P$, $h \notin R_n^*$, where

$$R_n^* = \{(i, j) \in \mathbb{Z}^2;\ -n \leqslant i,\ j \leqslant n\}, \qquad n \geqslant 1$$

By the Kolmogorow zero-one law we have

$$\bigcap_{n=1}^{\infty} \mathscr{B}_n = \mathscr{N}$$

Let $\varepsilon > 0$ be arbitrary. By the Doob martingale convergence theorem there exists n_0 such that

$$\|E^{\mathscr{B}_n} f\|^2 < 1 + \varepsilon \qquad \text{for} \quad n > n_0$$

If $g \in \Pi \backslash R_n$, $n > n_0$ then $\mathscr{A}_{-g} \subset \mathscr{B}_n$ and so

$$\| W^g f - 1\|^2 = \|E^{\mathscr{A}_{-g}} f\|^2 - 1 \leqslant \|E^{\mathscr{B}_n} f\|^2 - 1 < \varepsilon$$

This means that

$$W^g f \xrightarrow{s} 1 \qquad \text{as} \quad g \longrightarrow \infty$$

Remark 2. It is not clear whether any K-action of $\mathbb{Z}^2$ is ISR.

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